

# Hardy's function $Z(t)$ - results and problems

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**Abstract.** This is primarily an overview article on some results and problems involving the classical Hardy function

$$Z(t) := \zeta(\tfrac{1}{2} + it)(\chi(\tfrac{1}{2} + it))^{-1/2}, \quad \zeta(s) = \chi(s)\zeta(1-s).$$

In particular, we discuss the first and third moment of  $Z(t)$  (with and without shifts) and the distribution of its positive and negative values. A new result involving the distribution of its values is presented.

*AMS Mathematics Subject Classification* (2010): 11M06.

*Key Words and phrases:* Riemann zeta-function, Hardy's function, odd moments, distribution of values.

## 1 Definition of Hardy's function

The primary aim of this paper is to present some results and problems involving *Hardy's function*  $Z(t)$ , since in recent years there was a revival of interest in its study. This classical function (see e.g., the author's monograph [15] for an extensive account) has a century long history. It is defined as

$$(1.1) \quad Z(t) := \zeta(\tfrac{1}{2} + it)(\chi(\tfrac{1}{2} + it))^{-1/2},$$

where  $\chi(s)$  comes from the familiar functional equation for  $\zeta(s)$  (see e.g., Chapter 1 of [9]), namely  $\zeta(s) = \chi(s)\zeta(1-s)$  for  $s \in \mathbb{C}$ , so that

$$\chi(s) = 2^s \pi^{s-1} \sin(\tfrac{1}{2}\pi s) \Gamma(1-s), \quad \chi(s)\chi(1-s) = 1.$$

It follows that

$$\overline{\chi(\tfrac{1}{2} + it)} = \chi(\tfrac{1}{2} - it) = \chi^{-1}(\tfrac{1}{2} + it),$$

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so that  $Z(t) \in \mathbb{R}$  when  $t \in \mathbb{R}$ ,  $Z(t) = Z(-t)$ , and  $|Z(t)| = |\zeta(\frac{1}{2} + it)|$ . Thus the zeros of  $\zeta(s)$  on the “critical line”  $\Re s = 1/2$  correspond to the real zeros of  $Z(t)$ , which makes  $Z(t)$  an invaluable tool in the study of the zeros of the zeta-function on the critical line. Alternatively, if we use the symmetric form of the functional equation for  $\zeta(s)$ , namely

$$\pi^{-s/2} \zeta(s) \Gamma(\tfrac{1}{2}s) = \pi^{-(1-s)/2} \zeta(1-s) \Gamma(\tfrac{1}{2}(1-s)),$$

then for  $t \in \mathbb{R}$  we obtain

$$Z(t) = e^{i\theta(t)} \zeta(\tfrac{1}{2} + it), \quad e^{i\theta(t)} := \pi^{-it/2} \frac{\Gamma(\tfrac{1}{4} + \tfrac{1}{2}it)}{|\Gamma(\tfrac{1}{4} + \tfrac{1}{2}it)|} \quad (\theta(t) \in \mathbb{R}),$$

which implies that  $Z(t)$  is a smooth function.

For completeness, recall that the Riemann zeta-function is defined by

$$(1.2) \quad \zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_p (1 - p^{-s})^{-1}$$

for  $\Re s > 1$ , where  $p$  denotes primes. For other values of the complex variable  $s = \sigma + it$  ( $\sigma, t \in \mathbb{R}$ ) it is defined by analytic continuation. It is regular for  $s \in \mathbb{C}$ , except at  $s = 1$  where it has a simple pole with residue 1. The product representation in (1.2) shows that  $\zeta(s)$  does not vanish for  $\sigma > 1$ . The best known “zero-free region” for  $\zeta(s)$  is of the form

$$(1.3) \quad \sigma > 1 - C(\log t)^{-2/3}(\log \log t)^{-1/3} \quad (C > 0, t \geq t_0 > 0).$$

This was obtained in 1958 by the method of I.M. Vinogradov (see e.g., his works [34], [35], Chapter 4 of [22] and Chapter 6 of [9]). The best known numerical values in (1.3) are  $C = 1/57, 54$ ,  $t_0 = 4$ , and they are due to K. Ford [4].

## 2 Zeta-zeros on the critical line

Hardy’s original application [6] in 1914 of  $Z(t)$  was to show that  $\zeta(s)$  has infinitely many zeros on the critical line  $\Re s = 1/2$  (see e.g., E.C. Titchmarsh [23]). The argument is briefly as follows. Suppose on the contrary that, for  $T \geq T_0$ , the function  $Z(t)$  does not change sign. Then

$$(2.1) \quad \int_T^{2T} |Z(t)| dt = \left| \int_T^{2T} Z(t) dt \right|.$$

On one hand we have

$$\int_T^{2T} |Z(t)| dt = \int_T^{2T} |\zeta(\tfrac{1}{2} + it)| dt \geq \left| \int_T^{2T} \zeta(\tfrac{1}{2} + it) dt \right|.$$

One has the elementary formula (see e.g., Chapter 1 of [16])

$$(2.2) \quad \zeta(\tfrac{1}{2} + it) = \sum_{n \leq T} n^{-1/2-it} + \frac{T^{1/2-it}}{it - 1/2} + O(T^{-1/2}) \quad (T \leq t \leq 2T).$$

Using (2.2) it is easily found, on integrating termwise the right-hand side, that

$$\int_T^{2T} \zeta(\tfrac{1}{2} + it) dt = 2T + O(T^{1/2}).$$

This yields

$$(2.3) \quad \int_T^{2T} |Z(t)| dt = \int_T^{2T} |\zeta(\tfrac{1}{2} + it)| dt \gg T,$$

and the even slightly sharper lower bound (see K. Ramachandra [29])  $T(\log T)^{1/4}$  holds.

On the other hand, to bound the integral on the right-hand side of (2.1) we can use the approximate functional equation (this is a weakened form of the so-called Riemann–Siegel formula; for a proof see [9] or [33])

$$(2.4) \quad Z(t) = 2 \sum_{n \leq \sqrt{t/(2\pi)}} n^{-1/2} \cos\left(t \log \sqrt{\frac{t/(2\pi)}{n}} - \frac{t}{2} - \frac{\pi}{8}\right) + O\left(\frac{1}{t^{1/4}}\right).$$

If this expression is integrated and the second derivative test is applied (see [9] or [33]) it follows that

$$(2.5) \quad \int_T^{2T} Z(t) dt = O(T^{3/4}).$$

Thus from (2.1)–(2.5) we obtain

$$T \ll \int_T^{2T} |Z(t)| dt \ll T^{3/4},$$

which is a contradiction. This proves that  $\zeta(s)$  has infinitely many zeros on the critical line. In fact, the argument that leads to (2.5) actually shows that

$$N_0(T) \gg T^{1/4},$$

where  $N_0(T)$  denotes the number of complex zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$  for which  $\beta = 1/2, 0 < \gamma \leq T$ . Later Hardy refined his argument to show that  $N_0(T) \gg T$ .

A. Selberg (see [30] or [33] for a proof) improved this bound to  $N_0(T) \gg T \log T$ , which is one of the most important results of analytic number theory. In fact, this implies that

$$(2.6) \quad N_0(T) \geq CN(T)$$

for some  $C > 0$  and  $T \geq T_0 > 0$ . Here  $N(T)$  denotes the number of  $\rho = \beta + i\gamma$  for which  $0 < \gamma \leq T$ . One has the classical *Riemann–von Mangoldt formula* (see e.g., Chapter 1 of [9] for a proof)

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

and therefore (2.6) holds as a consequence of  $N_0(T) \gg T \log T$ .

N. Levinson [24] in 1974 showed that  $C = 1/3$  is permissible in (2.6), J.B. Conrey [2] 1989 obtained  $C = 2/5$ , that is, 40% of the zeta-zeros are on the critical line. The latest record was achieved by S. Feng [3], who proved that at least 41.73% of the zeros of  $\zeta(s)$  are on the critical line and at least 40.75% of those zeros are simple ( $\zeta(\rho) = 0 \Rightarrow \zeta'(\rho) \neq 0$ ) and on the critical line. Selberg's proof of (2.5) involved combining a “mollifier” to compensate for irregularities in the size of  $|\zeta(s)|$  and the method of Hardy (and Littlewood). Levinson introduced new ideas, and subsequent research refined on the existing methods.

**Notation.** Owing to the nature of this text, absolute consistency in notation could not be attained, although whenever possible standard notation is used. By  $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$  we denote the set of natural numbers, integers, real and complex numbers, respectively. The symbol  $\varepsilon$  will denote arbitrarily small positive numbers, not necessarily the same ones at each occurrence. The Landau symbol  $f(x) = O(g(x))$  and the Vinogradov symbol  $f(x) \ll g(x)$  both mean that  $|f(x)| \leq Cg(x)$  for some constant  $C > 0, g(x) > 0$  and  $x \geq x_0 > 0$ . By  $f(x) \ll_{a,b,\dots} g(x)$  we mean that the constant implied by the  $\ll$ -symbol depends on  $a, b, \dots$ . The symbol  $f(x) = \Omega_{\pm}(g(x))$  means that both  $\limsup_{x \rightarrow \infty} f(x)/g(x) > 0$  and  $\liminf_{x \rightarrow \infty} f(x)/g(x) < 0$  holds.

### 3 Moments of Hardy's function

#### 3.1 Discussion of $F_k(T)$

For  $k \in \mathbb{N}$  fixed, consider the  $k$ -th moment of  $Z(t)$ , namely the integral

$$(3.1) \quad F_k(T) := \int_0^T Z^k(t) dt.$$

Since  $|Z(t)| = |\zeta(\frac{1}{2} + it)|$ , it transpires that

$$F_{2k}(T) \equiv \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt,$$

which is one of the fundamental objects in the study of  $\zeta(s)$ . Even moments in general are a natural object of study, because of the elementary identity  $|z|^2 = z \cdot \bar{z}$ . When  $z = \zeta(\frac{1}{2} + it)^k$ , this permits one to develop the square and use various approximate functional equations etc. The reader is referred to the monographs of K. Ramachandra [29] and the author [10], which deal exclusively with mean values (moments) of  $\zeta(s)$ . Also the books of E.C. Titchmarsh [33] and the author [9] contain a lot of material on this subject, as does his review paper [16]. Thus only the study of  $F_{2k-1}(T)$  represents a novelty. The function  $Z(t)$  takes positive and negative values (and, heuristically, with a certain regularity), so that one expects there will be a lot of cancellations when one evaluates  $F_{2k-1}(T)$ . However, the following natural problem seems challenging in the general case.

**Problem 1.** *Show that, for  $k > 1$  a fixed integer, one has*

$$(3.2) \quad F_{2k-1}(T) = o\left(\int_0^T |\zeta(\frac{1}{2} + it)|^{2k-1} dt\right) \quad (T \rightarrow \infty).$$

The use of the Riemann–Siegel formula (2.4) does not seem adequate in proving (3.2). Although we have at our disposal *smooth variants* of this formula, which will be discussed a little later, the problem of establishing (3.2) nevertheless remains open. For a discussion involving problems with  $Z(t)$ , see the author's paper [14].

#### 3.2 Bounds for $F_1(t)$

We turn now to  $F(T) \equiv F_1(T)$ . In 2004 the author [12] improved (2.5) by obtaining a much stronger result than (3.2) for  $k = 1$ , namely

**Theorem 1.** *We have*

$$(3.3) \quad F(T) = \int_0^T Z(t) dt = O_\varepsilon(T^{1/4+\varepsilon}).$$

We sketch briefly the proof of (3.3). It is based (A.I. [10], 1990) on the use of a smooth approximate functional equation for  $Z^k(t)$ , namely

$$Z^k(t) = 2 \sum_{n \leq 2\tau} \rho\left(\frac{n}{\tau}\right) d_k(n) n^{-1/2} \cos\left(t \log \frac{\tau}{n} - \frac{k}{2}t - \frac{\pi k}{8}\right) + O(t^{\frac{k}{4}-1} \log^{k-1} t),$$

where for any fixed integer  $k \geq 1$ ,  $t \geq 2$ ,

$$\tau = \left(\frac{t}{\pi}\right)^{k/2},$$

and further notation is as follows. The generalized divisor function  $d_k(n)$  (generated by  $\zeta^k(s)$ , which makes it possible to define  $d_k(n)$  for an arbitrary  $k \in \mathbb{C}$ ) represents the number of ways  $n$  may be represented as the product of  $k$  factors ( $d_1(n) \equiv 1$ ,  $d_2(n) \equiv d(n)$ , the number of divisors of  $n$ ). The test function  $\rho(x)$  is a non-negative, smooth function supported in  $[0, 2]$ , such that  $\rho(x) = 1$  for  $0 \leq x \leq 1/b$  for a fixed constant  $b > 1$ , and  $\rho(x) + \rho(1/x) = 1$  for all  $x$ . The last condition induces a symmetry in the approximate functional equation for  $Z^k(t)$ .

For  $k = 1$  the error term gives  $O(T^{1/4})$  after integration. The integration of the main term produces exponential integrals which are evaluated by the classical saddle point method. There are a number of such results in the literature (see e.g., Chapter 2 of [9]). The one that is convenient is the following lemma (see p. 71 of the monograph by Karatsuba–Voronin [22]).

**Lemma 1.** *If  $f(x) \in C^{(4)}[a, b]$ ,  $f''(x) > 0$  in  $[a, b]$ , then*

$$\begin{aligned} \int_a^b e^{2\pi i f(x)} dx &= e^{\pi i/4} \frac{e^{2\pi i f(c)}}{\sqrt{f''(c)}} \\ &+ O(AV^{-1}) + O\left(\min(|f'(a)|^{-1}, \sqrt{A})\right) + O\left(\min(|f'(b)|^{-1}, \sqrt{A})\right), \end{aligned}$$

where the main term is to be halved if  $c = a$  or  $c = b$ , and

$$\begin{aligned} 0 < b - a &\leq V, \quad f'(c) = 0, \quad a \leq c \leq b, \\ f''(x) &\asymp A^{-1}, \quad f^{(3)}(x) \ll (AV)^{-1}, \quad f^{(4)}(x) \ll A^{-1}V^{-2} \quad (A > 0). \end{aligned}$$

Application of Lemma 1 and subsequent estimations and simplifications lead eventually to the upper bound in (3.3).

In [12] it was conjectured that

$$(3.4) \quad \int_0^T Z(t) dt = O(T^{1/4}), \quad \int_0^T Z(t) dt = \Omega_{\pm}(T^{1/4}).$$

This was proved, independently and by different methods, by M. Jutila [18], [19] and M.A. Korolev [23]. Therefore they established (up to the value of the numerical constants which are involved in the  $O$  and  $\Omega_{\pm}$  symbols) the true order of the integral in question. For the integral in (3.4) Korolev actually obtained the explicit bound

$$\left| \int_{2\pi}^T Z(t) dt \right| < 18.2T^{1/4} \quad (T \geq T_0).$$

### 3.3 The cubic moment of $Z(t)$

In what concerns the cubic moment  $F_3(T) \equiv \int_0^T Z^3(t) dt$ , in Oberwolfach 2003 I posed the following

**Problem 2.** *Does there exist a constant  $0 < c < 1$  such that*

$$(3.5) \quad F_3(T) \equiv \int_0^T Z^3(t) dt = O(T^c)?$$

*Perhaps even  $c = 3/4 + \varepsilon$  is permissible? Is  $c < 3/4$  impossible in (3.5)?*

To this day the problem remains open. However, if one considers the cubic moment of  $|Z(t)|$ , then it is known that

$$(3.6) \quad T(\log T)^{9/4} \ll \int_1^T |Z(t)|^3 dt = \int_1^T |\zeta(\tfrac{1}{2} + it)|^3 dt \ll T(\log T)^{9/4},$$

which establishes the true order of the integral in question. However, obtaining an asymptotic formula for this integral remains a difficult problem. The lower bound in (3.6) follows from general results of K. Ramachandra (see his monograph [29]), and the upper bound is a recent result of S. Bettin, V. Chandee and M. Radziwiłł [1].

In [15], equation (11.9), an explicit formula for the cubic moment of  $Z(t)$  was derived. This is

$$(3.7) \quad \int_T^{2T} Z^3(t) dt = 2\pi \sqrt{\frac{2}{3}} \sum_{(\frac{T}{2\pi})^{3/2} \leq n \leq (\frac{T}{\pi})^{3/2}} d_3(n) n^{-\frac{1}{6}} \cos(3\pi n^{\frac{2}{3}} + \tfrac{1}{8}\pi) + O_{\varepsilon}(T^{3/4+\varepsilon}),$$

where as usual  $d_3(n)$  is the divisor function

$$d_3(n) = \sum_{k\ell m=n} 1 \quad (k, \ell, m, n \in \mathbb{N}),$$

generated by  $\zeta^3(s)$  for  $\Re s > 1$ . Various techniques were used in [15] to estimate the exponential sum in (3.7), but nothing better than the weak  $O_\varepsilon(T^{1+\varepsilon})$  seems to come out.

A strong conjecture of the author is that

$$(3.8) \quad \int_1^T Z^3(t) dt = O_\varepsilon(T^{3/4+\varepsilon}).$$

Note that (3.8) would follow (by partial summation) from (3.7) and the bound

$$(3.9) \quad \sum_{N < n \leq N' \leq 2N} d_3(n) e^{3\pi i n^{2/3}} \ll_\varepsilon N^{2/3+\varepsilon}.$$

It may be remarked that the exponential sum in (3.9) is “pure” in the sense that the function in the exponential does not depend on any parameter as, for example, the sum

$$\sum_{N < n \leq N' \leq 2N} n^{it} = \sum_{N < n \leq N' \leq 2N} e^{it \log n} \quad (1 \leq N \ll \sqrt{t}),$$

which appears in the approximation to  $\zeta(\frac{1}{2} + it)$  (see e.g., Theorem 4.1 of [9]), depends on the parameter  $t$ . However, the difficulty in the estimation of the sum in (3.9) lies in the presence of the divisor function  $d_3(n)$  which, in spite of its simple appearance, is quite difficult to deal with.

Finally we note that not much can be said about  $F_{2k-1}(T)$  when  $k \geq 3$ . Even the conjecture in (3.2) of Problem 1 remains open.

### 3.4 Moments of $Z(t)$ with shifts

A related and interesting problem is to investigate integrals of  $Z(t)$  with “shifts”, i.e., integrals where one (or more) factor  $Z(t)$  is replaced by  $Z(t+U)$ . The parameter  $U$ , which does not depend on the variable of integration  $t$ , is supposed to be positive and  $o(T)$  as  $T \rightarrow \infty$ , where  $T$  is the order of the range of integration.

Some results on such integrals already exist in the literature. For example, R.R. Hall [5] proved that, for  $U = \alpha/\log T$ ,  $\alpha \ll 1$ , we have uniformly

$$(3.10) \quad \begin{aligned} \int_0^T Z(t)Z(t+U) dt &= \frac{\sin \alpha/2}{\alpha/2} T \log T + (2\gamma - 1 - 2\pi) T \cos \alpha/2 \\ &+ O\left(\frac{\alpha T}{\log T} + T^{1/2} \log T\right). \end{aligned}$$



Here  $\gamma = -\Gamma'(1) = 0.5772157\dots$  is Euler's constant. M. Jutila [20] obtained recently an asymptotic formula for the the integral in (3.10) when  $0 < U \ll T^{1/2}$ .

S. Shimomura [32] dealt with the quartic moment

$$(3.11) \quad \int_0^T Z^2(t)Z^2(t+U) dt,$$

under certain conditions on the real parameter  $U$ , such that  $(|U| + 1)/\log T \rightarrow 0$  as  $T \rightarrow \infty$ . When  $U \rightarrow 0+$ , Shimomura's expression for (3.11) reduces to

$$(3.12) \quad \int_0^T |\zeta(\tfrac{1}{2} + it)|^4 dt = \int_0^T Z^4(t) dt = \frac{1}{2\pi^2} T \log^4 T + O(T \log^3 T).$$

The (weak) asymptotic formula (3.12) is a classical result of A.E. Ingham [8] of 1928.

Finally we mention that the author [17] obtained an asymptotic formula for the integral of  $Z^2(t)Z(t+U)$ . This is formulated as

**Theorem 2.** *For  $0 < U = U(T) \leq T^{1/2-\varepsilon}$  we have, uniformly in  $U$ ,*

$$(3.13) \quad \int_{T/2}^T Z^2(t)Z(t+U) dt = O_\varepsilon(T^{3/4+\varepsilon}) + 2\pi\sqrt{\frac{2}{3}} \sum_{T_1 \leq n \leq T_0} h(n, U) n^{-1/6+iU/3} \exp(-3\pi i n^{2/3} - \pi i/8) \{1 + K(n, U)\}.$$

Here ( $d(n)$  is the number of divisors of  $n$ )

$$(3.14) \quad h(n, U) := n^{-iU} \sum_{\delta|n} d(\delta) \delta^{iU}, \quad T_0 := \left(\frac{T}{2\pi}\right)^{3/2}, \quad T_1 := \left(\frac{T}{\pi}\right)^{3/2},$$

$$(3.15) \quad K(n, U) := d_2 U^2 n^{-2/3} + \dots + d_k U^k n^{-2k/3} + O_k(U^{k+1} n^{-2(k+1)/3})$$

for any given integer  $k \geq 2$ , with effectively computable constants  $d_2, d_3, \dots$ .

The interval of integration is  $[T/2, T]$ , since if it is  $[0, T]$ , then  $K(n, u)$  is not necessarily small. Note that, as  $U \rightarrow 0+$ , the main term in (3.13) becomes the main term in (3.7). In other words, Theorem 2 is a generalization of (3.7). Therefore we may ask similar questions as was done in the case of  $F_3(T)$ .

**Problem 3.** *Is it true that there exists a constant  $0 < c < 1$  such that the integral in (3.13) is  $O(T^c)$  uniformly for  $0 < U = U(T) \leq T^{1/2-\varepsilon}$ ?*

The initial step in the proof of Theorem 2 is to write

$$(3.16) \quad \int_{T/2}^T Z^2(t)Z(t+U) dt = \frac{1}{i} \int_{1/2+iT/2}^{1/2+iT} \zeta^2(s)\zeta(s+iU)(\chi^2(s)\chi(s+iU))^{-1/2} ds.$$

The procedure of writing a real-valued integral like a complex integral is fairly standard in analytic number theory. For example, see the proof of Theorem 7.4 in E.C. Titchmarsh's monograph [33] on  $\zeta(s)$  and M. Jutila's recent work [20]. It allows one flexibility by suitably deforming the contour of integration in the complex plane. Incidentally, this method of proof is different from the proof of (3.7) in [15], which is based on the use of approximate functional equations.

In the complex integral in (3.16) we replace the segment of integration by  $[1 + \varepsilon + \frac{1}{2}iT, 1 + \varepsilon + iT]$ , and use the functional equation  $\zeta(s) = \chi(s)\zeta(1-s)$  etc. The problem is eventually reduced to the evaluation of exponential integrals whose saddle point satisfies (when  $U > 0$ ) a non-trivial cubic equation (i.e.,  $x^3 = ax + b$ ), whose solution is best found asymptotically. Lemma 1 is used for the evaluation of the ensuing saddle points, and Theorem 2 follows eventually.

## 4 The distribution of values of $Z(t)$

Let  $H = T^\theta$ ,  $0 < \theta \leq 1$ , and

$$(4.1) \quad I_+(T, H) := \int_{T, Z(t) > 0}^{T+H} Z(t) dt, \quad I_-(T, H) := \int_{T, Z(t) < 0}^{T+H} Z(t) dt.$$

Also let

$$(4.2) \quad \begin{aligned} \mathcal{J}_+(T, H) &:= \left\{ T < t \leq T+H : Z(t) > 0 \right\}, \quad \mathcal{K}_+(T, H) = \mu(\mathcal{J}_+(T, H)) \\ \mathcal{J}_-(T, H) &:= \left\{ T < t \leq T+H : Z(t) < 0 \right\}, \quad \mathcal{K}_-(T, H) = \mu(\mathcal{J}_-(T, H)), \end{aligned}$$

where  $\mu(\cdot)$  denotes measure. We are interested in bounding  $I_\pm, \mathcal{K}_\pm$ . In [14] the author proved that, unconditionally,

$$(4.3) \quad \begin{aligned} T(\log T)^{1/4} &\ll I_+(T, T) \ll T(\log T)^{1/4}, \\ T(\log T)^{1/4} &\ll -I_-(T, T) \ll T(\log T)^{1/4}. \end{aligned}$$

**Problem 4.** *Are there constants  $A_1, A_2 > 0$  such that*

$$I_+(T, T) = (A_1 + o(1))T(\log T)^{1/4}, \quad -I_-(T, T) = (A_2 + o(1))T(\log T)^{1/4} \quad (T \rightarrow \infty)?$$

We present now a new result, which is contained in

**Theorem 3.** *Let  $H = T^\theta$  with  $1/4 \leq \theta \leq 1$ . If the Riemann hypothesis is true, then for any number  $k > 1$  we have*

$$(4.4) \quad \begin{aligned} \mathcal{K}_+(T, H) &\gg_k H(\log T)^{-k/4} \\ \mathcal{K}_-(T, H) &\gg_k H(\log T)^{-k/4}, \end{aligned}$$

where  $\mathcal{K}_+(T, H), \mathcal{K}_-(T, H)$  are defined by (4.2).

**Proof of Theorem 3.** First note that in [14], [15] the author, for the left-hand sides in (4.4), obtained unconditionally the bounds  $T(\log T)^{-1/2}$  when  $H = T$ . The improvement in Theorem 3 is thus conditional, but the result holds in a much more general case.

Assume the Riemann hypothesis (all complex zeros of  $\zeta(s)$  have real parts equal to  $1/2$ ). K. Ramachandra [29] proved that

$$(4.5) \quad \int_T^{T+H} |Z(t)|^k dt \gg_k H(\log H)^{k^2/4} \quad (\log \log T \ll H \leq T, k \in \mathbb{R}, k \geq 0)$$

An explicit value for the constant implicit in the  $\gg$ -symbol in (4.5) is to be found in the work of M. Radziwiłł and K. Soundararajan [28]. As for the upper bound for the integral in (4.5), we have

$$(4.6) \quad \int_T^{T+H} |Z(t)|^k dt \ll_k H(\log H)^{k^2/4} \quad (T^\theta \ll H \leq T).$$

with  $H = T^\theta$  and  $0 < \theta \leq 1$ . This follows if one combines the results of A. Harper [7] and the author [13], both which are based on the method of K. Soundararajan [31]. Therefore it follows that

$$H(\log T)^{1/4} \ll \int_T^{T+H} |Z(t)| dt = \int_{\mathcal{J}_+(T, H)} Z(t) dt - \int_{\mathcal{J}_-(T, H)} Z(t) dt.$$

On the other hand,

$$\int_{\mathcal{J}_+(T, H)} Z(t) dt + \int_{\mathcal{J}_-(T, H)} Z(t) dt = \int_T^{T+H} Z(t) dt \ll T^{1/4}$$

by (3.4). Hence for  $H = T^\theta, 1/4 \leq \theta \leq 1$  we have

$$\begin{aligned} H(\log T)^{1/4} &\ll \int_{\mathcal{J}_+(T, H)} Z(t) dt, \\ H(\log T)^{1/4} &\ll - \int_{\mathcal{J}_-(T, H)} Z(t) dt. \end{aligned}$$

Suppose now that  $k$  satisfies  $1 < k < 2$ . By Hölder's inequality for integrals and (4.6) we have

$$\begin{aligned} H(\log T)^{1/4} &\ll \int_{\mathcal{J}_+(T, H)} Z(t) dt \leq \left( \int_{T, Z(t) > 0}^{T+H} Z^k(t) dt \right)^{1/k} \left( \mathcal{K}_+(T, H) \right)^{1-1/k} \\ &\ll_k (H(\log T)^{k^2/4})^{1/k} \left( \mathcal{K}_+(T, H) \right)^{1-1/k}. \end{aligned}$$

This gives

$$\begin{aligned} H^{(k-1)/k} (\log T)^{(1-k)/4} &\ll_k \left( \mathcal{K}_+(T, H) \right)^{(k-1)/k}, \\ \mathcal{K}_+(T, H) &\gg_k H(\log T)^{-k/4}. \end{aligned}$$

In a similar fashion it is found that

$$\mathcal{K}_-(T, H) \gg_k H(\log T)^{-k/4}.$$

This completes the proof of Theorem 3.

**Problem 5.** *Do there exist positive constants  $D_+, D_-$  such that*

$$\mathcal{K}_+(T, T) = (D_+ + o(1))T, \quad \mathcal{K}_-(T, T) = (D_- + o(1))T \quad (T \rightarrow \infty)?$$

*Is it true that  $D_+ = D_- = 1/2$ ?*

Of course, in general either  $\mathcal{K}_+(T, H) \gg H$  or  $\mathcal{K}_-(T, H) \gg H$  holds, but one cannot say which one of these lower bounds holds.

To continue our discussion on the evaluation of  $\mathcal{K}_+(T, T)$  (and  $\mathcal{K}_-(T, T)$ ), assume now the Riemann Hypothesis and the simplicity of zeta zeros. These very strong conjectures seem to be independent in the sense that it is not known whether either of them implies the other one. Then (since  $Z(0) = -1/2$ ) we have

$$(4.7) \quad \mathcal{K}_+(T, T) = \mu \left\{ T < t \leq 2T : Z(t) > 0 \right\} = \sum_{T < \gamma_{2n} \leq 2T} (\gamma_{2n} - \gamma_{2n-1}) + O(1),$$

where  $0 < \gamma_1 < \gamma_2 < \dots$  are the ordinates of complex zeros of  $\zeta(s)$ . Thus the problem of the evaluation is reduced to the evaluation of the sum in (4.7). It seems reasonable that the differences  $\gamma_{2n} - \gamma_{2n-1}$  and  $\gamma_{2n+1} - \gamma_{2n}$  are evenly distributed, which heuristically indicates that  $D_+$  exists and that  $D_+ = D_- = 1/2$ . However, proving this is hard.

Finally, to conclude our discussion on the distribution of values of  $Z(t)$ , note that the sum in (4.7) is related to the sum ( $\alpha \geq 0$  is fixed)

$$\sum_{\alpha}(T) := \sum_{\gamma_n \leq T} (\gamma_n - \gamma_{n-1})^{\alpha},$$

which was investigated in [11]. The sum  $\sum_{\alpha}(T)$  in turn can be connected to the Gaussian Unitary Ensemble hypothesis (see A.M. Odlyzko [26], [27]) and the pair correlation conjecture of H.L. Montgomery [25]. Both of these conjectures assume the Riemann Hypothesis and e.g., the former states that, for

$$0 \leq \alpha < \beta < \infty, \quad \delta_n = \frac{1}{2\pi}(\gamma_{n+1} - \gamma_n) \log \left( \frac{\gamma_n}{2\pi} \right),$$

we have

$$\sum_{\gamma_n \leq T, \delta_n \in [\alpha, \beta]} 1 = \left( \int_{\alpha}^{\beta} p(0, u) du + o(1) \right) \frac{T}{2\pi} \log \left( \frac{T}{2\pi} \right) \quad (T \rightarrow \infty).$$

Here  $p(0, u)$  is a certain probabilistic density, given by complicated functions defined in terms of prolate spheroidal functions. In fact, in [11] the author proved that, if the RH and the Gaussian Unitary Ensemble hypothesis hold, then for  $\alpha \geq 0$  fixed and  $T \rightarrow \infty$ ,

$$\sum_{\alpha}(T) = \left( \int_0^{\infty} p(0, u) u^{\alpha} du + o(1) \right) \left( \frac{2\pi}{\log \left( \frac{T}{2\pi} \right) - 1} \right)^{\alpha-1} T.$$

Also note that, since  $\Re \log \zeta(\frac{1}{2} + it) = \log |Z(t)|$ , a classical result of A. Selberg (see [30], Vol. 1) gives, for any real  $\alpha < \beta$ ,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \mu \left\{ t : t \in [T, 2T], \alpha < \frac{\log |Z(t)|}{\sqrt{\frac{1}{2} \log \log T}} < \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-\frac{1}{2}x^2} dx,$$

but here we are interested in the distribution of values of  $Z(t)$  and not  $|Z(t)|$ . Recently J. Kalpokas and J. Steuding [21] proved that for  $\phi \in [0, \pi)$ ,

$$(4.8) \quad \sum_{0 < t \leq T, \zeta(\frac{1}{2} + it) \in e^{i\phi} \mathbb{R}} \zeta(\frac{1}{2} + it) = \left( 2e^{i\phi} \cos \phi \right) \frac{T}{2\pi} \log \frac{T}{2\pi e} + O_{\varepsilon}(T^{1/2+\varepsilon}),$$

and an analogous result holds for the sums of  $|\zeta(\frac{1}{2} + it)|^2$ . It is unclear whether (4.8) and the other approaches mentioned above can be put to use in connection with our problems.

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